Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Semester 1 2016-2017 Exam Solutions

All the exam questions are unseen.

1. (a) (i) $\frac{3}{7} - \frac{4}{9} = \frac{(3)(9) - (7)(4)}{(7)(9)} = \frac{-1}{63} = -\frac{1}{63}$.

(ii) $-\frac{2}{7} \times \left(-\frac{5}{7}\right) = \frac{(-2)(-5)}{(7)(7)} = \frac{10}{49}$.

(iii) $\frac{2}{9} \div \frac{11}{5} = \frac{2}{9} \times \frac{5}{11} = \frac{(2)(5)}{(9)(11)} = \frac{10}{99}$.

(iv) $-6^2 = -\left(6^2\right) = -36$.

(v) $\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \frac{1}{\left(\frac{16}{81}\right)^{\frac{3}{4}}} = \frac{1}{\left(\left(\frac{16}{81}\right)^{\frac{1}{4}}\right)^3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{8/27} = \frac{27}{8}$.

(vi) $6 \div (7 - (-9) \times (-8)) = 6 \div (7 - 72) = 6 \div (-65) = -\frac{6}{65}$.

(vii) Since $4^3 = 64$, it follows that $\log_4 64 = 3$.

(viii) Since $3^{-3} = \frac{1}{27}$, it follows that $\log_3 \frac{1}{27} = -3$. **[8]**

(b) (i) $x^6 \times x^{-8} = x^{6+(-8)} = x^{-2}$.

(ii) $x^{\frac{1}{2}} \div x^{-\frac{2}{3}} = x^{\frac{1}{2}-\left(-\frac{2}{3}\right)} = x^{\frac{1}{2}+\frac{2}{3}} = x^{\frac{7}{6}}$.

(iii) $\left(x^{-2}\right)^{-3} = x^{-2(-3)} = x^6$. **[3]**

(c)

$$\begin{aligned}
\log_a\left(\left(\frac{y^3}{x^4}\right)^{-2}\right) &= -2\log_a\left(\frac{y^3}{x^4}\right) \\
&= -2\left[\log_a\left(y^3\right) - \log_a\left(x^4\right)\right] \\
&= -2\left[3\log_a y - 4\log_a x\right] \\
&= 8\log_a x - 6\log_a y.
\end{aligned}$$

[2]

(d) (i) $9.94999 = 9.9$ to one decimal place.

(ii) $0.0004454 = 0.00045$ to two significant figures.

(iii) $132410.01 = 1.3241001 \times 10^5$ in scientific notation.

(iv) $0.000249 = 2 \times 10^{-4}$ in scientific notation to one significant figure. **[4]**

(e) $(3x^2 - 2x + 3) - (-3x - 3) = 3x^2 + (-2x + 3x) + (3 + 3) = 3x^2 + x + 6$. **[1]**

(f)

$$\begin{aligned}(2x^4-3x^2)(-3x^2+4) &= (2x^4)(-3x^2+4)+(-3x^2)(-3x^2+4)\\ &= (2x^4)(-3x^2)+(2x^4)(4)+(-3x^2)(-3x^2)+(-3x^2)(4)\\ &= -6x^{4+2}+8x^4+9x^{2+2}-12x^2\\ &= -6x^6+8x^4+9x^4-12x^2\\ &= -6x^6+17x^4-12x^2.\end{aligned}$$

[2]

(g)

$$\begin{array}{r|l} & \quad\;\; x+2 \\ \hline x+3\,) & x^2+5x+2 \\ & -x^2-3x \\ \hline & \qquad 2x+2 \\ & \quad\; -2x-6 \\ \hline & \qquad\;\; -4 \end{array}$$

This tells us that $\dfrac{x^2+5x+3}{x+3}=(x+2)+\dfrac{-4}{x+3}$.
So the quotient is $x+2$ and the remainder is -4. **[4]**

(h) $\displaystyle\sum_{i=-3}^{2} -i^3 = -(-3)^3-(-2)^3-(-1)^3-0^3-1^3-2^3 = 27+8+1+0-1-8 = 27.$ **[2]**

(i) $\dbinom{7}{3} = \dfrac{7\times 6\times 5}{3\times 2} = 35.$ **[2]**

(j)

$$\begin{aligned}(2x-3y)^3 &= (2x)^3+\binom{3}{1}(2x)^2(-3y)+\binom{3}{2}(2x)(-3y)^2+(-3y)^3\\ &= 8x^3-36x^2y+54xy^2-27y^3.\end{aligned}$$

[4]

2. (a) **[2]**

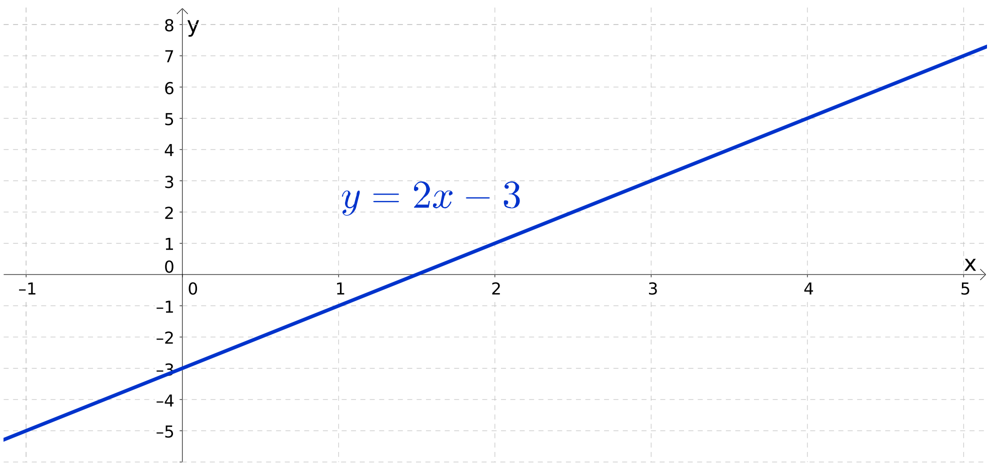

Figure 1: Graph of the line $y = 2x - 3$.

(b) If we add three times the second equation to two times the first we obtain

$$\begin{array}{rrrrrrr} & -6x & + & 8y & = & 22 \\ + & 6x & + & -9y & = & -24 \\ \hline & & & -y & = & -2 \end{array}$$

Hence $y = 2$ and on substituting this into the second equation we get $2x - 3(2) = -8$, so that $2x = -8 + 6 = -2$ and hence $x = -1$.
Thus the solution is $x = -1$ and $y = 2$. **[3]**

(c) Using $(x_1, y_1) = (-1, -2)$ and $(x_2, y_2) = (2, 3)$, the formula tells us that the midpoint of the line segment is

$$\left(\frac{-1+2}{2}, \frac{-2+3}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

[1]

3. (a)

$$\begin{aligned} 2x^2 - 3x + 1 &= 2\left\{x^2 - \frac{3}{2}x + \frac{1}{2}\right\} \\ &= 2\left\{\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{1}{2}\right\} \\ &= 2\left\{\left(x - \frac{3}{4}\right)^2 - \frac{1}{16}\right\} \\ &= 2\left(x - \frac{3}{4}\right)^2 - \frac{1}{8}. \end{aligned}$$

[3]

(b) In this case $a = 2$, $b = -3$ and $c = 1$.
Hence the solutions of the equation are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)} \\ &= \frac{3 \pm \sqrt{9-8}}{4} \\ &= \frac{3 \pm \sqrt{1}}{4} \\ &= \frac{3 \pm 1}{4} \\ &= \frac{1}{2} \text{ or } 1. \end{aligned}$$

[2]

(c) From Part (b) we know that the graph cuts the x-axis when $x = \dfrac{1}{2}$ and when $x = 1$.
Next, when $x = 0$, $y = 1$, so the graph cuts the y-axis when $y = 1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right) = \left(-\frac{-3}{2(2)}, -\frac{(-3)^2 - 4(2)(1)}{4(2)}\right) = \left(\frac{3}{4}, -\frac{1}{8}\right).$$

We now have all the information we need and I have sketched the graph in Figure 2.

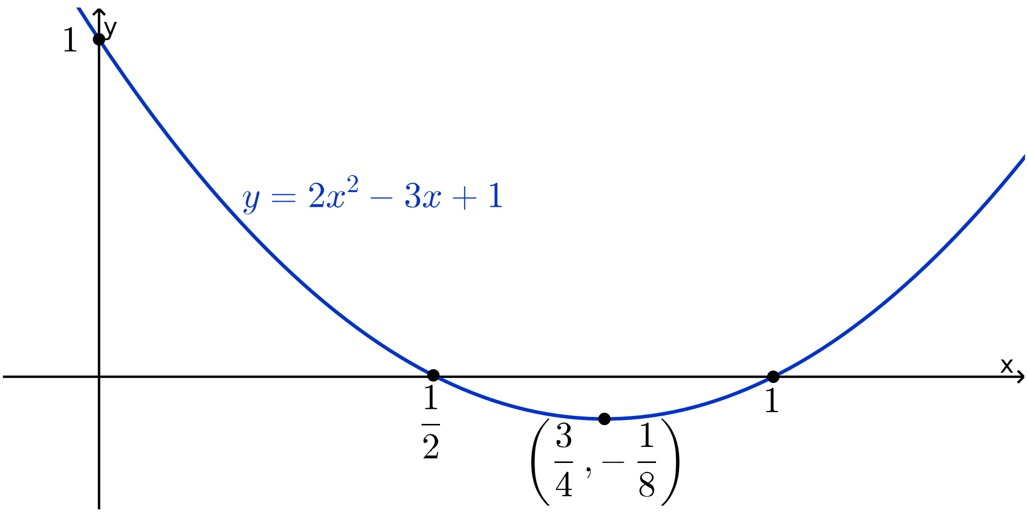

Figure 2: The Graph of the function $y = 2x^2 - 3x + 1$.

[4]

4. (a) (i) This is not a function.
For example, $f(-1)$ is not defined, since $-2(-1) - 1 = 1$ does not lie in the codomain.

(ii) This is a function.
Its domain is $\mathbb{R}^-$ and its codomain is $\mathbb{R}^+$.

[4]

(b) Figure 3 shows the graph of the function

$$\begin{aligned} f\colon \{-4, -2, 0, 1, 3\} &\to \{-3, -2, 0, 2, 3\} \\ -4 &\mapsto 2 \\ -2 &\mapsto -2 \\ 0 &\mapsto 2 \\ 1 &\mapsto 0 \\ 3 &\mapsto 3 \end{aligned}$$

[2]

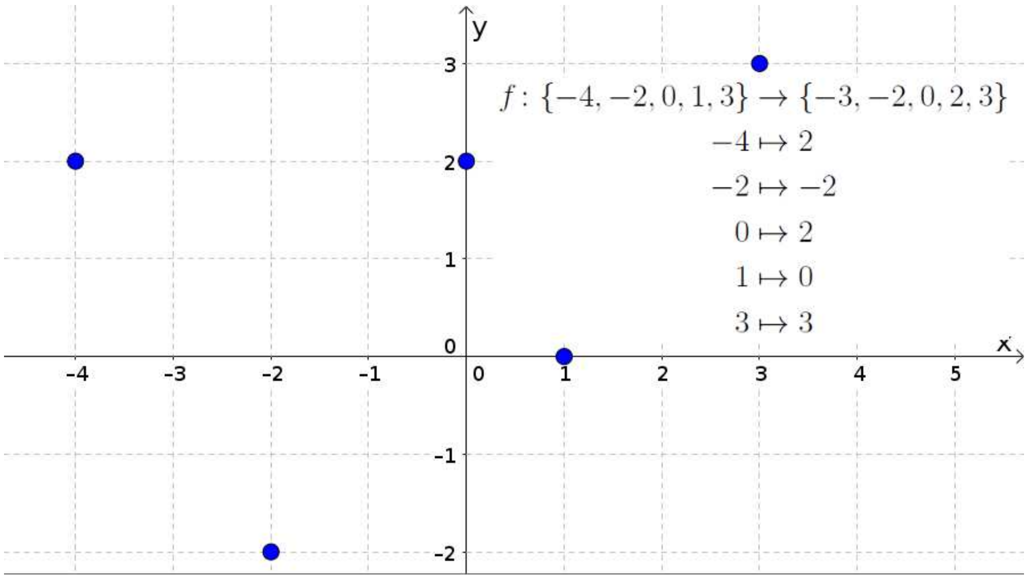

Figure 3: The graph of the function defined in Question 4(b).

(c) The function h crosses the x-axis so it must be a log function. In addition h decreases as x increases, so it can't be (iii) and so must be (v).
Next g and k lie below the x-axis, so they must be (ii) and (vi). Now $y = 2^x$ increases as x increases, so $y = -2^x$ decreases as x increases. On the other hand $y = \left(\frac{2}{7}\right)^x$ decreases as x increases, so $y = -\left(\frac{2}{3}\right)^x$ increases as x increases. Thus g must be (ii) and k must be (vi).
Finally l lies above the x-axis, so must be (i) or (iv). However $y = \left(\frac{2}{7}\right)^x$ decreases as x increases, so it can't be l. Thus l must be (i).
Summarizing: g is (ii), h is (v), k is (vi) and l is (i). **[4]**

(d) (i) This function is not injective since $f(1) = B = f(4)$.
It is not surjective since there is no x with $f(x) = C$.
It is not bijective since it is neither injective nor surjective.

(ii) This function is injective.
It is not surjective since there is no x with $f(x) = 0$.
It is not bijective since it is not surjective. **[3]**

(e) Neither of the functions in Part (d) are bijective, so neither of them have an inverse.

[1]

5. (a) $105° = 105 \times \dfrac{\pi}{180} = \dfrac{7\pi}{12}$ Radians. **[1]**

(b) $\dfrac{7\pi}{4}$ Radians $= \left(\dfrac{7\pi}{4} \times \dfrac{180}{\pi}\right)^{\circ} = 315°$. **[1]**

(c) In this case we want to find $\tan(\theta)$ when $\theta = \dfrac{-2\pi}{3}$.

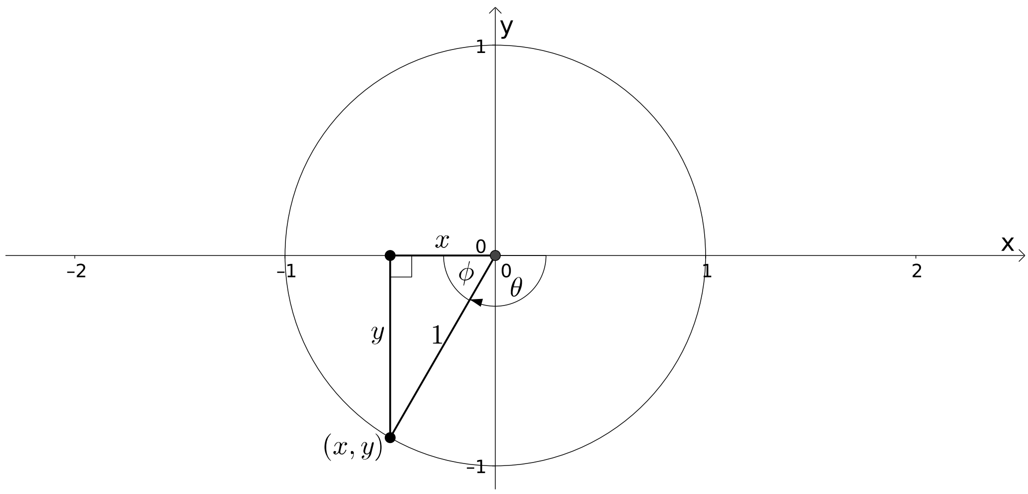

Figure 4: Calculation of $\tan\left(\dfrac{-2\pi}{3}\right)$.

Looking at Figure 4, we see that we need to find $\dfrac{y}{x}$, since this is by definition $\tan\left(\dfrac{-2\pi}{3}\right)$. Now, also from Figure 4, $\phi = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using the table of common values, $\tan(\phi) = \sqrt{3}$. But also by definition $\tan(\phi) = \dfrac{|y|}{|x|}$. Looking at the diagram, we see that x and y are negative, so $\dfrac{y}{x} = \dfrac{|y|}{|x|}$. Hence $\tan\left(\dfrac{-2\pi}{3}\right) = \sqrt{3}$. **[3]**

(d) (i) We will first use $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ with $A = \pi$ and $B = \dfrac{\pi}{4}$.

We have $\sin\left(\pi + \dfrac{\pi}{4}\right) = \sin(\pi)\cos\left(\dfrac{\pi}{4}\right) + \cos(\pi)\sin\left(\dfrac{\pi}{4}\right)$.

Now $\sin(\pi) = 0$ and $\cos(\pi) = -1$ and we can use our table of common values to see that $\sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.

Hence $\sin(\pi)\cos\left(\dfrac{\pi}{4}\right) + \cos(\pi)\sin\left(\dfrac{\pi}{4}\right) = 0 \cdot \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}(-1) = -\dfrac{1}{\sqrt{2}}$.

Thus $\sin\left(\dfrac{5\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$.

(ii) We will first use the fact that $\tan(-\theta) = -\tan(\theta)$.

Thus $\tan\left(-\dfrac{\pi}{12}\right) = -\tan\left(\dfrac{\pi}{12}\right)$.

Next, to find $\tan\left(\dfrac{\pi}{12}\right)$, we will use $\tan^2(\theta) = \dfrac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$ with $\theta = \dfrac{\pi}{12}$.

Hence $\tan^2\left(\dfrac{\pi}{12}\right) = \dfrac{1 - \cos\left(\dfrac{\pi}{6}\right)}{1 + \cos\left(\dfrac{\pi}{6}\right)}$.

However, using the table of common values, $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$.

Thus $\tan^2\left(\frac{\pi}{12}\right) = \dfrac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}} = \dfrac{2-\sqrt{3}}{2+\sqrt{3}}$ and $\tan\left(\frac{\pi}{12}\right) = \sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}$, and hence $\tan\left(-\frac{\pi}{12}\right) = -\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}$. **[4]**

(e) If we solve $b^2 = a^2 + c^2 - 2ac\cos(B)$ for $\cos(B)$ we obtain $\cos(B) = \dfrac{a^2+c^2-b^2}{2ac}$ and on substituting for a, b and c, $\cos(B) = \dfrac{8^2+5^2-7^2}{2(8)(5)} = \dfrac{1}{2}$. Hence $B = 60°$. **[3]**

6. (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)^2-2x^2}{h} \\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-2x^2}{h} \\ &= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-2x^2}{h} \\ &= \lim_{h\to 0}\frac{4xh+2h^2}{h} \\ &= \lim_{h\to 0} 4x+2h \\ &= 4x. \end{aligned}$$

[2]

(b) (i) $f'(x) = 0$.

(ii) $f'(x) = 4x^{4-1} = 4x^3$.

(iii) $f'(x) = -3(-\sin(-3x)) = 3\sin(-3x)$.

(iv) $f'(x) = \dfrac{1}{2}\cos\left(\dfrac{1}{2}x\right)$.

(v) $f'(x) = -4\left(-\dfrac{1}{4}x^{-\frac{1}{4}-1}\right) - 3\left(-2e^{-2x}\right) - 3\left(\dfrac{1}{x}\right) = x^{-\frac{5}{4}} + 6e^{-2x} - \dfrac{3}{x}$. **[6]**

7. (a) $\displaystyle\int 1\,dx = x + c$. **[1]**

(b) $\displaystyle\int_{-1}^{1} x^4\,dx = \left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{1}{5}(1^5) - \frac{1}{5}(-1)^5 = \frac{2}{5}$. **[2]**

(c)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(2x)\ dx &= \left[\frac{1}{2}(-\cos(2x))\right]_0^{\frac{\pi}{2}}\\ &= \left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}}\\ &= -\frac{1}{2}\cos(\pi) - \left(-\frac{1}{2}\cos(0)\right)\\ &= -\frac{1}{2}(-1) + \frac{1}{2}(1)\\ &= 1\end{aligned}$$

[2]

(d)

$$\begin{aligned}\int e^{-2x} - x^{-\frac{4}{5}}\ dx &= \frac{1}{-2}e^{-2x} - \frac{1}{-\frac{4}{5}+1}x^{-\frac{4}{5}+1} + c\\ &= -\frac{1}{2}e^{-2x} - \frac{1}{1/5}x^{\frac{1}{5}} + c\\ &= -\frac{1}{2}e^{-2x} - 5x^{\frac{1}{5}} + c\end{aligned}$$

[2]

8. (a) (i) The mean is $\overline{x} = \frac{1}{9}(0+3+3+(-6)+4+6+0+2+(-3)) = \frac{9}{9} = 1$.

(ii) The list in ascending order is $-6, -3, 0, 0, 2, 3, 3, 4, 6$.
Since there are nine numbers (an odd number), the median is
$m = x_{\frac{9+1}{2}} = x_5 = 2$.

(iii) There are two zeros, two threes and one of each of the other numbers, so the modes are 0 and 3.

(iv) Since we have an odd number of numbers, we discard the median and split the remaining numbers into a lower half $-6, -3, 0, 0$ and an upper half $3, 3, 4, 6$. There are four numbers in each of these new groups (an even number), so in each case the median is $\frac{x_{\frac{4}{2}} + x_{\frac{4}{2}+1}}{2} = \frac{x_2 + x_3}{2}$. Thus the lower quartile is $Q1 = \frac{-3+0}{2} = -\frac{3}{2}$ and the upper quartile is $Q3 = \frac{3+4}{2} = \frac{7}{2}$. Hence the interquartile range is $Q_3 - Q_1 = \frac{7}{2} - \left(-\frac{3}{2}\right) = 5$.

[5]

(b) There are five points, so $n = 5$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{5} x_i = -4 + (-2) + 0 + 3 + 5 = 2$$
$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{5} y_i = 3 + 1 + 1 + (-2) + (-5) = -2$$

$$\begin{aligned}\sum_{i=1}^{n} x_i y_i =& \sum_{i=1}^{5} x_i y_i \\ =&(-4)(3) + (-2)(1) + (0)(1) + (3)(-2) + (5)(-5) \\ =& -12 - 2 + 0 - 6 - 25 \\ =& -45.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} x_i^2 =& \sum_{i=1}^{5} x_i^2 \\ =&(-4)^2 + (-2)^2 + 0^2 + 3^2 + 5^2 \\ =&16 + 4 + 0 + 9 + 25 \\ =&54.\end{aligned}$$

Hence

$$\begin{aligned}m &= \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} \\ &= \frac{5(-45) - (2)(-2)}{5(54) - 2^2} \\ &= \frac{-221}{266} \\ &= -\frac{221}{266},\end{aligned}$$

and

$$c = \overline{y} - m\overline{x} = \frac{\sum_{i=1}^{5} y_i}{5} - m\frac{\sum_{i=1}^{5} x_i}{5} = \frac{-2}{5} - \left(-\frac{221}{266}\right)\frac{2}{5} = -\frac{2}{5} + \frac{221}{665} = -\frac{9}{133}.$$

Thus the line of best fit is $y = -\dfrac{221}{266}x - \dfrac{9}{133}$. **[7]**